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On Global Maximum of a Convex Terminal Functional in Optimal Control Problems

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Abstract. In this paper necessary and sufficient conditions (related to Pontryagin's principle) for a global maximum of a convex terminal functional for different types of control systems are proved. A few examples are given.

Key words: Convex function, Pontryagin's principle, optimality conditions, local and global solutions.

1. Introduction

This paper considers some nonconvex optimal control problems for the system of ordinary differential equations. Such problems arise, for example, in applications previously described [1–3].

Unfortunately, sufficient optimality conditions such as those in the dynamic programming method [4, 7, 8] and Krotov's conditions, [4, 9] have some disadvantages which are not characteristic of Pontryagin's maximum principle. First of all, it refers to the numerical methods constructed on their basis [4–9]. For instance, it is sufficient to consider the proofs of convergence for numerical methods, and the results of numerical experiments [5, 6, 8, 9].

The present paper proposes an approach to construct necessary and sufficient global maximum conditions for a convex terminal functional based on earlier papers [10–13]. For this class of problems Pontryagin's maximum principle follows from the general global optimality condition. First, a brief proof of global maximum conditions for a convex function on a set from R^n is given.

Second, the maximization problem of a convex terminal functional for the linear system is considered.

Third, optimal control for a semi-linear system with a convex objective functional and terminal equality and inequality constraints is investigated.

Finally, the problem of optimal control of a general system with a convex terminal functional and local Lipschitz equality and inequality constraints is considered.

All the results are illustrated with examples. The brackets $\langle \cdot, \cdot \rangle$ denote the scalar product in the space R^n or a bilinear form value on the product of Banach spaces $X^* \times X$, that are in duality [4, 14–18].

Standard notations of convex and nonlinear analysis [14–18] are used in this paper.

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2. The Maximization Problems

Consider the following optimization problem

$$f(x) \to \max, \quad x \in D, \tag{1}$$

where $f(\cdot)$ is a convex lower semicontinuous proper functional [14–19] on a reflexive Banach space X, and D is an arbitrary subset from X:

$$D \subset \text{int dom } f. \tag{2}$$

THEOREM 1. *If $z \in D$ is a global maximum point of Problem (1) ($z \in Argmax(1)$), then*

$$\left.\begin{aligned} \forall y : f(y) = f(z), \ \ \forall y^* \in \partial f(y), \\ \langle y^*, x - y \rangle \leqslant 0, \ \ \forall x \in D. \end{aligned}\right\} \tag{E.1}$$

If, besides,

$$\exists v \in X : -\infty < f(v) < f(z) < +\infty, \tag{H}$$

then the optimality condition (E.1) becomes sufficient for $z \in Argmax\,(1)$ as well.

Proof. (1) The necessity is obvious.

(2) Let there exist $u \in D : f(u) > f(z)$. Consider the convex closed set

$$S(f, z) = \{x \in X | f(x) \leqslant f(z)\}.$$

This is obvious because (H) int $S(f, z) \neq 0$ and $u \notin S(f, z)$. Thus, in virtue of reflexivity of X [4, 14]

$$\exists y, f(y) = f(z) : \|y - u\| = \inf\{\|x - u\| \,|\, x \in S(f, z)\} > 0. \tag{3}$$

And from the extremum theory [4, 14–18] it follows that the point $y \in$ Argmin(ϕ, $S(f, z)$), $\phi(x) = \|x - u\|$ can be characterized by the following condition:

$$\begin{aligned} \exists x^* \in \partial\phi(y), \quad \exists y^* \in \partial f(y), \quad \lambda_0 \geqslant 0, \quad \exists \lambda \geqslant 0, \\ \lambda_0 x^* + \lambda y^* = 0, \quad \lambda_0 + \lambda > 0. \end{aligned} \tag{4}$$

Note, by definition [14]

$$\partial\phi(y) = \{x^* \in X^* \,|\, \|x^*\| = 1, \langle x^*, y - u \rangle = \|y - u\|\}. \tag{5}$$

If $\lambda_0 = 0$ then $\lambda > 0$, and the condition $y \in \text{Argmin}(f, X)$ follows from (4). This is impossible because of (H) and the equality $f(y) = f(z)$. If, now, $\lambda = 0$ then because of (4)

$$0 \in \partial\phi(y).$$

The latter is also impossible because of (3).

Then, after dividing (4) by $\lambda > 0$, we have

$$\alpha_0 x^* + y^* = 0, \quad \alpha_0 > 0, x^* \in \partial\phi(y), \quad y^* \in \partial f(y).$$

From the above fact and on account of (3) and (5) we have

$$\langle y^*, u - y\rangle = \alpha_0\langle x^*, y - u\rangle = \alpha_0\|y - u\| > 0,$$

that is in contradiction with (E.1). □

NOTES. (1) It is easily seen that assumption (H) is essential for sufficiency, since, for example, for the case $X = R^n$ and a differentiable function $f(\cdot)$, all the points of the global minimum f on R^n satisfy the condition $f'(y) = 0$, and, therefore, trivially satisfy condition (E.1).

(2) The well-known optimality condition for a differentiable function f

$$\langle f'(z), x - z\rangle \leqslant 0, \quad \forall x \in D, \tag{6}$$

usually proved for a convex set D, follows from condition (E.1) when $y = z$ ($X = R^n$). The convexity of D is not necessary here. Convexity of f is required instead.

EXAMPLE 1. Let in problem (1) $X = R$, R – the real axis,

$$f(x) = (x^2 - 2), \quad D = [-2, -0.2] \cup [0, 1].$$

The classical optimality condition (6) is satisfied in the two points $z_1 = 1, z_2 = -2$. In this case for $y_1 = -1$, $f(y_1) = f(z_1) = -1$ and $u = -1.5$, the condition (E.1) is violated

$$\langle f'(y_1), u_1 - y_1\rangle = -2 \cdot (-0.5) > 0.$$

Hence, z_1 is not the global solution. For z_2 there exists the unique point $y_2 = 2$: $f(y_2) = f(z_2)$. Moreover, it can be readily seen that

$$\langle f'(y_2), x - y_2\rangle \leqslant 0, \quad \forall x \in co(D),$$

thus, $z_2 \in \text{Argmax}(f, D)$.

Note that in the general case the verification of condition (E.1) is a rather complicated process. To simplify it one can rewrite (E.1) in a somewhat different form.

THEOREM 2. *Let the conditions of Theorem 1 be satisfied. Then, to satisfy the condition* $z \in Argmax(1)$ *it is necessary that*

$$\text{(E.2)} \begin{cases} \forall y : f(y) = f(z), \quad \forall y^* \in \partial f(y) \\ \textit{any maximizing sequence } \{x^k\} \textit{ of the problem} \\ \langle y^*, x \rangle \to \max, \quad x \in D & (7) \\ \textit{satisfies the condition} \\ \lim_{k\to\infty} \langle y^*, x^k - y \rangle \leqslant 0. & (8) \end{cases}$$

If (H) *is true, these conditions are also sufficient.*

3. Control of the Linear System

Let the notation $\mathring{\forall}$ mean "for almost every". Consider the following control problem:

$$g(x(t_1)) \to \max, \tag{9}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + c(t), \tag{10}$$

$$x(t_0) = x^0,$$

$$u(\cdot) \in \mathcal{U} = \{u \in L^r_\infty(T) | u(t) \in U, \ \mathring{\forall} t \in T\}, \tag{11}$$

where $t \in [t_0, t_1] \triangleq T$, $x(t) = (x_1(t), \ldots, x_n(t))$, $u(t) = (u_1(t), \ldots, u_r(t))$; $x^0 \in R^n$ – the initial state; the moments t_0, t_1 are fixed; $A(t)$, $B(t)$, $c(t)$ are $(n \times n)$, $(n \times r)$ and $(n \times 1)$ matrices with the entries from $L_\infty(T)$ respectively; U is a compact set from R^n, and the function $g : R^n \to R$ (R – the real axis) is convex and differentiable.

It is known [1, 4, 14] that each control $u(\cdot) \in \mathcal{U}$ gives the unique solution $x(t, u)$ of the Cauchy problem (10), that is an absolutely continuous function (with respect to t) satisfying equations (10) almost everywhere on $T(\mathring{\forall} t \in T)$. This fact will further be denoted as $x(\cdot, u) \in H^1(T)$.

Let D be the reachable set of system (10), (11) at the moment t_1 [1, 4, 5]:

$$D = \{p \in R^n | p = x(t_1, u), u \in \mathcal{U}\}. \tag{12}$$

It is known [4, 5] under the assumption made that the set D is convex and compact.

Now, the problem (9)–(11) can be rewritten in the form:

$$g(p) \to \max, \ p \in D. \tag{13}$$

Assume that

$$\exists p \in D, \exists q \in R^n : -\infty < g(q) < g(p). \tag{H.1}$$

Then, obviously, condition (H) holds for problem (13).

Due to the compactness of D and the convexity (and, consequently, continuity) of $g(\cdot)$, problem (13) has the solution $z = x(t_1, w) \in D$, $w \in \mathcal{U}$.

The feasible control $w(\cdot) \in \mathcal{U}$, generating the point z, is the global optimal in problem (9)–(11).

Moreover, according to Theorem 2, $z \in$ Argmax (13) iff

$$\text{(E.3)} \begin{cases} \forall y \in R^n, g(y) = g(z), \\ \text{the solution } x_0(y) \text{ of the following convex problem} \\ \langle g'(y), x \rangle \to \max, \quad x \in D, \qquad (14) \\ \text{(which exists due to the compactness of the set } D\text{) satisfies the} \\ \text{following inequality} \\ \langle g'(y), x_0(y) - y \rangle \leqslant 0. \qquad (15) \end{cases}$$

Now, this optimality condition can be restated in terms of problem (9)–(11).

For this purpose $\forall y \in R^n : g(y) = g(z)$ we introduce the conjugate state $(\psi(t, y) = \psi(t) = \psi_1(t), \ldots, \psi_n(t))$ as follows:

$$\dot{\psi} = -A^*\psi, \quad \psi(t_1) = -g'(y). \tag{16}$$

It is known [4, 5, 14] that problem (16) has a unique absolute continuous solution $\psi(\cdot, y) \in H^1(T)$.

Let $u(\cdot) \in \mathcal{U}$ be a feasible control, $x(\cdot) = x(\cdot, u)$ – the corresponding phase trajectory. It may be easily seen that the following conditions [1, 4, 5, 6] hold

$$\left. \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\langle \psi, x \rangle = \langle \psi, Bu + c \rangle, \quad \mathring{\forall} t \in T; \\ &-\langle g'(y), x(t_1, u) \rangle = \langle \psi(t_0), x(t_0) \rangle + \\ &\qquad \int_T \langle \psi(t), B(t)u(t) + c(t) \rangle \, \mathrm{d}t. \end{aligned} \right\} \tag{17}$$

Then, due to the construction, the feasible control $u_0(\cdot) \in \mathcal{U}$ will be optimal in problem (14) iff

$$\langle g'(y), x(t_1, u_0) - x(t_1, u) \rangle \geqslant 0, \quad \forall u \in \mathcal{U}.$$

Hence, in virtue of (17) we have

$$\int_T \langle \psi(t), B(t)[u(t) - u_0(t)] \rangle \, \mathrm{d}t \geqslant 0, \quad \forall u \in \mathcal{U}.$$

As known from [18, 19], due to the local structure of the set U the latter inequality is equivalent to the following condition.

$$\langle \psi(t), B(t)[v - u_0(t)] \rangle \geqslant 0, \ \ \forall v \in U, \ \ \mathring{\forall} t \in T. \tag{18}$$

To put this in another way, it is necessary and sufficient that the following minimum condition be satisfied:

$$\langle \psi(t), B(t)u_0(t) \rangle = \min_{v \in U} \langle \psi(t, y), B(t)v \rangle, \ \ \mathring{\forall} t \in T. \tag{19}$$

for providing optimality of the control $u_0(\cdot, y)$ in problem (14).

So, let $x_0(\cdot, y) = x(\cdot, u_0(\cdot, y))$ be the solution of the system (10) that corresponds to the control $u_0(\cdot, y) \in \mathcal{U}$ satisfying the minimum condition (19) with the function $\psi(\cdot) = \psi(\cdot, y)$ as a solution for system (16). Having taken into account all these notations we proved the following result.

THEOREM 3. *The control $w(\cdot) \in \mathcal{U}$ is the global optimal in problem (9)–(11) iff*

$$\text{(E.4)} \begin{cases} \forall y \in R^n, g(y) = g(x(t_1, w)) \\ \textit{the inequality} \\ \langle g'(y), x_0(t_1, y) - y \rangle \leqslant 0 \\ \textit{holds.} \end{cases} \tag{20}$$

Now let us show that the well-known maximum principle by L.S. Pontryagin is the particular case of the optimality condition (E.4).

COROLLARY. *It follows from condition* (E.4) *of Theorem 3 at $y = z = x(t_1, w)$, that the global optimal control $w(\cdot) \in \mathcal{U}$ satisfies the minimum condition*

$$\langle \psi(t), B(t)w(t) \rangle = \min_{v \in U} \langle \psi(t), B(t)v(t) \rangle, \tag{21}$$

where $\psi(t)$ is the solution of the system

$$\dot{\psi} = -A^*(t)\psi, \quad \psi(t_1) = -g'(x(t_1, w)). \tag{22}$$

Proof. Put $y = z = x(t_1, w)$ in (E.4). Then, due to the linearity of the system (10) and the functional in (14), the control $u_0(\cdot, z)$, satisfying the minimum condition (19) at $y = z$, gives the global maximum to the following control problem:

$$\langle g'(z), x(t_1) \rangle \to \max,$$

$$\dot{x} = Ax + Bu + c, x(t_0) = x^0, \quad u \in \mathcal{U}. \tag{23}$$

Let us demonstrate, that for the state $x_0(\cdot)$, corresponding to the control $u_0(\cdot, z)$, the equality

$$\langle g'(z), x_0(t_1) \rangle = \langle g'(z), z \rangle \tag{24}$$

holds. Actually, on the one hand, the process $(z(\cdot), w(\cdot))$, for which $z(t) = x(t, w)$, $z(t_1) = z$ is feasible, and thus

$$\langle g'(z), x_0(t_1)\rangle \geqslant \langle g'(z), z\rangle.$$

On the other hand, in virtue of (E.4) for $y = z = z(t_1)$, the converse inequality holds and, hence, equality (24) holds.

Hence, the process $(z(\cdot), w(\cdot))$ is the solution for the problem (23) and, therefore, satisfies Pontryagin's maximum principle as well. □

Let us verify the validity of the condition (E.4) using the following examples.

EXAMPLE 1. Consider the control problem

$$g(x(t_1)) = x^2(t_1) \to \max$$

$$\dot{x} = u, \quad x(0) = 1, \quad t \in T = [0, 2]$$

$$-1 \leqslant u(t) \leqslant 1, \quad \mathring{\forall} t \in T.$$

Obviously, the control $u_1(t) \equiv -1$ satisfies Pontryagin's maximum principle:

$$x_1(t) = x(t, u_1) = 1 - t, \quad x_1(2) = -1,$$

$$\psi_1(t) \equiv -g'(x_1(2)) = 2,$$

$$2u_1(t) = \min_v \{2v | v \in [-1, 1]\}, \quad \mathring{\forall} t \in T.$$

However, this control does not provide the global maximum for the considered problem.

Actually, for $y_1 = 1$, $g(y_1) = g(x_1(t_1)) = 1$ we have

$$\psi(t, y_1) \equiv -g'(y_1) = -2.$$

Then, from the minimum condition

$$\psi(t, y_1)u_0(t) = \min_v \{-2v | v \in [-1, 1]\}$$

we have $u_0(t) \equiv 1$ and, correspondingly,

$$x_0(t) = x(t, u_0) = 1 + t, \quad x_0(t_1) = 3.$$

Hence,

$$\langle g'(y_1), x_0(t_1) - y_1\rangle = 2(3 - 1) > 0,$$

and therefore, the control $u_1 \equiv -1$ is not global optimal.

Let us show that the control $u_0(t) \equiv 1$ is global optimal.

Indeed, there exist only two points $y_0 = 3 = x_0(t_1)$, $y = -3$ such that

$$g(y) = g(x_0(t_1)).$$

Verify condition (E.4) at these two points.

$$\psi_0(t) = \psi(t, y_0) = -g'(y_0) = -6.$$

Hence, it follows that the control $u_0(\cdot)$ satisfies Pontryagin's maximum principle: $\overset{\circ}{\forall} t \in T$

$$\psi_0(t)u_0(t) = \min_v\{\psi_0(t)v | v \in [-1, 1]\}.$$

Furthermore, due to the equality $x_0(t_1) = y_0$ we have

$$\langle g'(y_0), x_0(t_1) - y_0\rangle = 0.$$

Then, for $y = -3$, $\psi(t, y) \equiv -g'(y) = 6$, and from the maximum condition:

$$\overset{\circ}{\forall} t \in T \quad \psi(t, y)u(t, y) = \min\{\psi(t, y)v | v \in [-1, 1]\}$$

we have $u(t, y) \equiv u_1(t) \equiv -1$. And then

$$\langle g'(y), x(t_1) - y\rangle = (-6)(-1 + 3) < 0.$$

Therefore, the control $u_0(t) \equiv 1$ is global optimal.

EXAMPLE 2. Let $g(x) = x_1^2 + x_2^2$, $x \in R^2$. Consider the following control problem:

$$g(x(t_1)) \to \max, \quad t \in T = [0, 2],$$

$$\dot{x}_1 = x_2, \quad x_1(0) = 2,$$

$$\dot{x}_2 = u, \quad x_2(0) = -1,$$

$$-1 \leqslant u(t) \leqslant 1, \quad \overset{\circ}{\forall} t \in T.$$

It is known from [1–6, 14] that such problems can be considered as those of object's controlled motion (x_1 – object's position, x_2 – object's speed, u – acceleration). In this case, the goal of $g(x(t_1))$ maximization can be interpreted as the maximal deviation from the point 0 at a maximum velocity at the moment $t_1 = 2$.

In view of the object's initial position $x_1(0) = 2$, we can assume that the motion will be continued in the positive direction, i.e., with the control $u(t) \equiv 1$. Hence, $x_2(t) = -1 + t$, $x_2(2) = 1$, $x_1(t) = 2 + (t^2/2) - t$, $x_1(2) = 2$. Then, it is easily seen that the conjugate system has the following solution:

$$\psi_1(t) \equiv \psi_1(t_1) = -4, \quad \psi_2(t) = -10 + 4t.$$

From the minimum condition

$$\langle \psi(t), B(t)u(t)\rangle = \psi_2(t)u(t) = \min\{\psi - 2(t)v | v \in [-1, 1]\}, \quad \mathring{\forall} t \in T,$$

we obtain the control $u(t) \equiv 1$, which, consequently, satisfies Pontryagin's maximum principle. This fact increases the hopes on optimality of $u(t) \equiv 1$.

Nevertheless, taking the point $y = (-1, -2)$ for which $g(y) = g(x(t_1, u)) = 5$, we obtain the following situation. Since $g'(y) = (-2, -4)$, the conjugate system has the solution:

$$\psi_1(y, t) = \psi_1(t_1) = 2, \qquad \psi_2(y, t) = 8 - 2t.$$

Then from the minimum condition: $\mathring{\forall} t \in T$

$$\psi_2(y, t)u(y, t) = \min\{\psi_2(y, t)v | v \in [-1, 1]\}$$

we obtain the control $u(y, t) \equiv -1$. The following vector-function is the corresponding solution for the control system:

$$x_2(y, t) = -1 - t, \quad x_2(y, t_1) = -3,$$

$$x_1(y, t) = 2 - (t^2/2) - t, \quad x_1(y, t_1) = -2.$$

Now, let us verify condition (E.4):

$$\langle g'(y), x(y, t_1) - y\rangle = \langle(-2, -4), (-2, 0)\rangle > 0.$$

Hence, the optimality condition (E.4) does not hold for the control $u(t) \equiv 1$ and, so, it is not global optimal.

Going back to the mechanical interpretation of this control problem, we may conclude that braking expenditures appeared to be so great that these expenditures annihilated the advantages of the initial position taken by the object under control.

4. Maximization of a Convex Nonsmooth Terminal Functional in the Semi-Linear System with State Constraints

Consider the following control problem:

$$g_0(x(t_1)) \to \max, \quad t \in T = [t_0, t_1], \tag{25}$$

$$\dot{x}(t) = A(t)x(t) + f(u(t), t), \quad x(t_0) = x^0 \tag{26}$$

$$g_1(x(t_1)) \leqslant 0, \quad i = 1, \ldots, m \tag{27}$$

$$\Lambda(x(t_1)) = b \in R^k, \tag{28}$$

where $g_l : R^n \to R\,(i = 0, \dots, m, R$ –real axis) are convex, not necessarily differentiable functions; the matrix $A(\cdot)$ is the same as in (10); $f(u,t)$ is the Caratheodory function, i.e., continuous with respect to the first variable and measurable w.r.t. the second one.

Furthermore, the mapping $\Lambda : \mathbb{R}^n \to \mathbb{R}^k$ $(k < n)$ is linear and regular:

$$\Lambda(R^n) = R^k. \tag{29}$$

Finally, there is the constraint (11) on the control $u(\cdot)$ in the problem (25)–(28). Introducing, as in Section 2, the reachable set

$$D = \{p \in R^n | p = x(t_1, u), \;\; u \in \mathcal{U}\},$$

where $x(t,u)$ is the solution of the Cauchy problem (26), that corresponds to the control $u(\cdot) \in \mathcal{U}$, we can reformulate the problem (11), (25)–(28) as follows

$$\left.\begin{array}{l} g_0(p) \to \max, \quad p \in D \\ g_l(p) \leqslant 0, \quad i = 1, \dots, m, \quad \Lambda p = b, \end{array}\right\} \tag{30}$$

In virtue of the above assumptions, the reachable set is convex and compact according to Lyapunov's theorem and the image convexity lemma [5, 14], and therefore, the feasible set of problem (30) is convex and compact.

Let assumption (H.1) of Section 2 hold.

Then, as follows from Theorem 2, the point $z \in D$ provides the global maximum for the problem (30) iff

$$\text{(E.5)} \begin{cases} \forall y \in R^n, g_0(y) = g_0(z), \quad \forall y^* \in \partial g_0(y), \\ \text{the solution } x_0(y, y^*) \text{ (which exists due to the compactness of the} \\ \text{feasible set in the problem (30)) of the following extremum problem} \\ \left.\begin{array}{l} \langle y^*, p\rangle \to \max, \quad p \in D, \\ g_i(p) \leqslant 0, \quad i = 1, \dots, m, \quad \Lambda p = b, \end{array}\right\} \quad (31) \\ \text{satisfies the condition:} \\ \langle y^*, x_0(y, y^*) - y\rangle \leqslant 0. \end{cases}$$

□

Note that due to the assumptions made above, problem (31) is convex. Hence, as is known from [4, 14–18], the necessary (in case of $\lambda_0 > 0$ sufficient) condition of global maximum in problem (31) is presented by the following optimality condition:

$$\left.\begin{array}{c} x_0(y) = x_0(y, y^*) \\ \exists(\lambda, \mu) \neq 0 \in R^{m+k+1}, \\ \lambda_i \geqslant 0, \quad i = 0, 1, \dots, m, \quad \mu \in R^k, \\ \lambda_i g_i(x_0(y)) = 0, \quad i = 1, \dots, m, \\ \exists y_i^* \in \partial g_i(x_0(y)), \quad i = 1, \dots, m, \end{array}\right\} \tag{32}$$

$$\left\langle -\lambda_0 y^* + \sum_1^m \lambda_i y_i^* + \Lambda^* \mu, p - x_0(y) \right\rangle \geqslant 0, \quad \forall p \in D. \tag{33}$$

It is known from [4, 14–18] that equality $\lambda_0 = 1$ holds when Slater's condition holds.

Let us transform condition (33) on account of the structure of the problem under consideration. To do this $\forall y : g_0(y) = g_0(z)$, $\forall y^* \in \partial g_0(y)$ we introduce the conjugate state

$$\psi(t; y, y^*) = \psi(t) = (\psi_1(t), \ldots, \psi_n(t))$$

as the solution for the equality system

$$\left.\begin{aligned} \dot{\psi} &= -A^*(t)\psi \\ \psi(t_1) &= -\lambda_0 y^* + \textstyle\sum_1^m \lambda_i y_i^* + \Lambda^* \mu. \end{aligned}\right\} \tag{34}$$

Denote by $x_0(\cdot, y) = x(\cdot, u_0, y, y^*)$ the state trajectory corresponding to the control $u_0(\cdot) \in \mathcal{U}$, for which $x(t_1, u_0, y, y^*) = x_0(y)$. Then condition (33) has the following form

$$\langle \psi(t_1, y, y^*), x(t_1, u) - x_0(t_1, u_0) \rangle \geqslant 0, \quad \forall u \in \mathcal{U}. \tag{35}$$

Taking into account the obvious equality

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t), x(t, u) \rangle = \langle \psi(t), f(u(t), t) \rangle,$$

it is easily seen, that [4–6]

$$\begin{aligned} \langle \psi(t_1), x(t_1, u) \rangle &= \langle \psi(t_0), x(t_0, u) \rangle + \int_T \frac{\mathrm{d}}{\mathrm{d}t} \langle \psi(t), x(t, u) \rangle \, \mathrm{d}t \\ &= \langle \psi(t_0), x^0 \rangle + \int_T \langle \psi(t), f(u(t), t) \rangle \, \mathrm{d}t \end{aligned}$$

Using these relations, condition (35) may be rearranged in the following form:

$$\int_T \langle \psi(t), f(u(t), t) - f(u_0(t), t) \rangle \, \mathrm{d}t \geqslant 0, \quad \forall u \in \mathcal{U}.$$

Due to the local structure of the set $\mathcal{U}$,

$$\langle \psi(t), f(u_0(t), t) \rangle = \min_v \{ \langle \psi(t), f(v, t) \rangle | v \in U \}, \quad \mathring{\forall} t \in T. \tag{36}$$

Let $x_0(\cdot, u_0, y, y^*)$ be the solution of the Cauchy problem (26) for $u_0 = u_0(y, y^*)$, satisfying the minimum condition (36) with the function $\psi = \psi(\cdot, y, y^*)$ which is the solution for system (34) with the vector $(\lambda, \mu)(y, y^*)$ satisfying (32). Now we can formulate the following result.

THEOREM 4. *The control $w(\cdot)$ is global optimal in the problem (11), (25)–(28) iff*

$$(\mathrm{E.6})\begin{cases} \forall y \in R^n : g_0(y) = g_0(x(t_1, w)), \quad \forall y^* \in \partial g_0(y) \\ \text{the inequality} \\ \langle y^*, x_0(t_1, y, y^*) - y\rangle \leqslant 0 \\ \text{holds.} \end{cases}$$

□

NOTES. (1) If there exists the control $\bar{u} \in U$ such that, for the corresponding state $\bar{x}(\cdot) = x(\cdot, \bar{u})$ the Slater's condition

$$g_i(\bar{x}(t_i)) < 0, \quad i = 1, \ldots, m,$$

$$\Lambda\bar{x}(t_1) = b$$

holds, one can put $\lambda_0 = 1$ in (34).

(2) Obviously $\forall y : g_0(y) = g_0(x(t_1, w))$, $\forall y^* \in \partial g_0(y)$ there exists, in the general case, the proper set of Lagrange multipliers

$$(\lambda, \mu)(y, y^*) = (\lambda_0, \lambda_1, \ldots, \lambda_m, \mu) \in R^{m+k+1}$$

for the corresponding problem (31).

(3) In the same way as in Section 3, one can prove that the global optimal control $w(\cdot)$ satisfies the Pontryagin's maximum principle (36) with $\psi(t)$ as the solution for the system (34) when $y = x(t_1, w)$, $y^* \in \partial g_0(x(t_1, w))$.

EXAMPLE 3. Consider the problem

$$g_0(x(t_1)) = x_1^2(t_1) + x_2^2(t_1) \to \max,$$

$$g_1(x(t_1)) = (x_1(t_1) + 1)^2 - 4 \leqslant 0,$$

$$\dot{x}_1 = x_2, \quad x_1(0) = -1,$$

$$\dot{x}_2 = u^2 - 2u, \quad x_2(0) = 0,$$

$$-2 \leqslant u(t) \leqslant 2, \quad \mathring{\forall} t \in T = [0, 2].$$

Consider the control $u_0(t) \equiv 1$. The corresponding phase trajectory is

$$x_2(t) = -t, \quad x_2(2) = -2, \quad x_1(2) = -3,$$

$$x_1(t) = -1 - t^2/2, \quad g_0(x(u_0, t_1)) = 13.$$

Then we verify whether the control $u_0(t)$ satisfies the maximum principle. Indeed, when $\lambda = 0.5$ we have

$$\psi_1(t) \equiv \psi_1(t_1) = -2x_1(t_1) + \lambda(x_1(t_1) + 1) = 5,$$

$$\psi_2(t_1) = 4, \quad \psi_2(t) = 14 - 5t.$$

It is easily seen, that the control $u_0(t) \equiv 1$ satisfies the minimum condition: $\overset{\circ}{\forall} t \in T$

$$\begin{aligned}\langle\psi(t), f(u(t), t)\rangle &= \psi_2(t)[u^2(t) - 2u(t)] \\ &= \min_v\{(14 - 5t)(v^2 - 2v) | v \in [-2, 2]\},\end{aligned}$$

and, hence, it satisfies Pontryagin's maximum principle also. Nevertheless, let us show that it is not global optimal.

To do this consider the point $y = (3, 2)$,

$$g_0(y) = g_0(x(u_0, t_1)) = 13.$$

Construct the conjugate system corresponding to y

$$\psi_1(y, t) \equiv \psi_1(y, t_1) = -2y_1 + 2\mu(y_1 + 1) = 8\mu - 6,$$

$$\psi_2(y, t_1) = -2y_2 = -4.$$

Set $\tau = 2 - 2\sqrt{2}/3$, $\mu = (8 - 3\tau)/4(2 - \tau) = 3(1 = \sqrt{2})/(4\sqrt{2}) > 1$. In this case

$$\psi_2(y, t) = -4 + \int_t^2 (8\mu - 6)\,\mathrm{d}\theta = (6 - 8\mu)t + 16\mu - 16.$$

It is obvious that

$$\psi_2(y, t) \begin{cases} > 0, & \forall t \in [0, \tau], \\ = 0, & t = \tau, \\ < 0, & \forall t \in [\tau, 2], \end{cases} \tag{37}$$

Thus, from the minimum condition: $\overset{\circ}{\forall} t \in T$

$$\psi_2(y, t)[u^2(t) - 2u(t)] = \min_v\{\psi_2(y, t)(v^2 - 2v) | v \in [-2, 2]\}, \tag{38}$$

we come to the conclusion that the control $u(y, \mu; t)$ has the following form

$$u_*(t) = u(y, \mu, t) = \begin{cases} 1, & t \in [0, \tau] \\ -2, & t \in [\tau, 2] \end{cases} \tag{39}$$

Note that finding the minimum in (38) can be realized with the use of Theorems 1 and 2.

Next, find the state $x_*(t) = x(y, \mu, t)$ corresponding to the control $u_*(t)$.

For $t \in [0, \tau]$ we have, as above,

$$\begin{array}{ll} x_2(t) = -t, & x_2(\tau) = -\tau, \\ x_1(t) = -1 - t^2/2, & x_1(\tau) = -1 - \tau^2/2. \end{array}$$

If $t \in [\tau, 2]$, then

$$\dot{x}_2(t) \equiv 8, \quad x_2(t) = 8t - 9\tau, \quad x_2(2) = 16 - 9\tau,$$

$$x_1(t) = 4t^2 - 9\tau t + 4.5\tau^2 - 1, \quad x_1(2) = 1.$$

Finally, verify the optimality condition (E.7):

$$0.5\langle g'(y), x_*(t_1) - y\rangle = 22 - 18\tau = 12\sqrt{2} - 14 > 0.$$

Therefore, the control $u_0(t) \equiv 1$ is not global optimal.

5. A General Optimal Control Problem

In this section we study the following problem:

$$g_0(x(t_1)) \to \max, \quad t \in T = [t_0, t1], \tag{40}$$

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(t_0) = x^0, \tag{41}$$

$$g_i(x(t_1)) \leqslant 0, \quad i = 1, \dots, m, \tag{42}$$

$$g_i(x(t_1)) = 0, \quad i = m + 1, \dots, k, \tag{43}$$

where the control $u(t) \in L^r_\infty(T)$ is subject to the constraint (11), $g_0 : R^n \to R \cup \{+\infty\}$ (R – real axis) is a convex and not necessarily smooth proper function, $g_i : R^n \to R$, $i = 1, \dots, k$ are local Lipschitz functions [17].

Let us further assume that $f(x, u, t)$ is a function continuous with respect to $(x, u) \in R^{n+r}$ and measurable with respect to t, which has the derivative $f_x(x, u, t)$ with the same properties. We assume also that for any feasible control $u(\cdot) \in \mathcal{U}$ the function $f(x, u(t), t)$ is integrable on T $\forall x \in R^n$. Then, as known from [4, 14], the Cauchy problem is solvable. Introduce, as above, the reachable set:

$$D = \{p \in R^n | p = x(t_1, u), \ u \in \mathcal{U}\},$$

where $x(\cdot, u)$ is the solution of the Cauchy problem (41), corresponding to the feasible control $u(\cdot) \in \mathcal{U}$.

In the further consideration we assume that the set D is compact. The Lipschitz condition for $f(x, u, t)$ on (x, u) [4, 5] is sufficient for that. Then problem (11), (40)–(43) can be represented in the following form:

$$\begin{array}{l} g_0(p) \to \max, \quad p \in D, \\ g_i(p) \leqslant 0, \quad i = 1, \dots, m, \\ g_i(p) = 0, \quad i = m + 1, \dots, k. \end{array} \tag{44}$$

Due to the assumption made above, there exists a solution of this problem if the feasible set is not empty.

Assume that assumption (H.1) holds for the function $g_0(\cdot)$. Then, according to Theorems 1 and 2, the point $z \in D$ gives the global maximum in problem (44) iff

(E.7)
$$\begin{cases} \forall y \in R^n : g_0(y) = g_0(z), \quad \forall y^* \in \partial g_0(y), \\ \text{any stationary point } x_0 = x_0(y, y^*) \text{ (i.e., which verifies local optimality conditions) in the following extremum problem:} \\ \langle y^*, p \rangle \to \max, \quad p \in D, \\ g_i(p) \leqslant 0, \quad i = 1, \ldots, m, \qquad (45) \\ g_i(p) = 0, \quad i = m+1, \ldots, k, \\ \text{satisfies the condition} \\ \langle y^*, x_0(y, y^*) - y \rangle \leqslant 0. \qquad (46) \end{cases}$$

NOTES. (1) As already noted, a solution and all stationary points of the problem (45) depend on $y : g_0(y) = g_0(z)$, and $y^* \in \partial g_0(y)$.

(2) According to above assumptions, the problem (45) is not convex in the general case, but this problem is "better" than the problem (44) whose nonconvexity generated by the objective functional does not exist any longer, since the objective functional in (45) is linear.

Now, let us apply Pontryagin's maximum principle to the problem (45). Then, in order that the control $u_0(\cdot)$ be optimal (may be locally optimal) it is necessary that in problem (45)

(E.8)
$$\begin{cases} \exists (\lambda_0, \lambda_1, \ldots, \lambda_k) \neq 0 \in R^{k+1}, \\ \lambda_i \geqslant 0, \quad i = 0, 1, \ldots, m, \\ \lambda_i g_i(x_0(t_1)) = 0, \quad i = 1, \ldots, m, \\ \exists y_1^* \in \partial_c g_i(x_0(t_1)), \quad i = 1, \ldots, k, \end{cases} \qquad (47)$$

for which $\overset{\circ}{\forall} t \in T$

$$H(\psi(t), x_0(t), u_0(t), t) = \max_{v \in U} H(\psi(t), x_0(t), v, t) \qquad (48)$$

where $x_0(\cdot) = x(\cdot, u(\cdot)), x_0(t_1) = x_0(y, y^*)$,

$$H(\psi, x, u, t) = \langle \psi, f(x, u, t) \rangle,$$

and the function $\psi(\cdot) \in H^1(T)$ is the solution for the following system of equalities:

$$\left.\begin{aligned} \dot{\psi}(t) &= -f_x^*(x_0(t), u_0(t), t)\psi(t), \\ \psi(t_1) &= \lambda_0 y^* - \sum_1^k \lambda_i y_i^*. \end{aligned}\right\} \tag{49}$$

Thus the following theorem is proved.

THEOREM 5. *In order that control $w \in \mathcal{U}$ be globally optimal in problem (11), (40)–(43) it is necessary and (if assumption (H.1) holds) sufficient that*

$$\text{(E.9)} \begin{cases} \forall y \in R^n : g_0(y) = g_0(x(t_1, w)), \quad \forall y^* \in \partial g_0(y) \\ \textit{the condition} \\ \langle y^*, x_0(t_1, y, y^*) - y\rangle \leqslant 0, \qquad (50) \\ \textit{holds for any control } u_0(\cdot, y, y^*) = u_0(\cdot), \textit{ satisfying condition (48)} \\ \textit{with the function } \psi(\cdot, y, y^*) \textit{ which is the solution of (49). Here,} \\ x_0(\cdot, y, y^*) = x(\cdot, u_0(\cdot, y, y^*)) \textit{ is the solution of system (41) that} \\ \textit{corresponds to the control } u_0(\cdot, y, y^*). \end{cases}$$

Note that, generally speaking, condition (E.8) is not sufficient for the process $\{x_0(\cdot), u_0(\cdot)\}$ to provide the global maximum in problem (45).

But, as noted above, problem (45) linearized w.r.t. the objective functional is simpler than the initial problem (11), (40)–(43). Roughly speaking, as far as the objective function is concerned, the nonconvexity degree of problem (11), (40)–(43) is reduced.

The above examples show that this fact facilitates the way out of the "local holes".

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